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Finally attention may be drawn to Problem 391 in an interesting collection by Ralph A. Roberts.*

10. It would be interesting, did space permit, to treat the nodal cubics (the reciprocals of tricuspidal quartics), whose equations may be thrown into the form†

$$X^{\frac{1}{3}} + Y^{\frac{1}{3}} + Z^{\frac{1}{3}} = 0,$$

by a method similar to that which we have used in the foregoing pages.

The results of the present paper were obtained four years ago. Within the past eighteen months a book by A. B. Basset‡ has appeared, making special note of the value of the method of projection in connection with the study of higher plane curves, whereas previously this theory had been largely confined to the treatment of conics.

SACKVILLE, N. B.
APRIL, 1903.

NOTE ON A PARTIAL DIFFERENTIAL EQUATION OF THE FIRST ORDER§

BY E. D. ROE, JR.

THIS note contains an elementary proof of the following familiar theorem :‖

A necessary and sufficient condition that a function $f(x_1, x_2, \ldots, x_n)$ *be a solution of the equation*

$$\frac{\partial f}{\partial x_1} + \frac{\partial f}{\partial x_2} + \cdots + \frac{\partial f}{\partial x_n} = 0 \tag{1}$$

is that f *be a function of the differences of the* x*'s.*

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* *A Collection of Examples and Problems on Conics and some of the Higher Plane Curves*, Dublin, 1882.

† Salmon, *loc. cit*, p. 184.

‡ *An Elementary Treatise on Cubic and Quartic Curves*, Cambridge, 1901.

§ Presented to the American Mathematical Society at the Evanston meeting, September 2–3, 1902

‖ *Cf.* Faà di Bruno's *Binäre Formen*, p 29 and p. 46, where, however, only the sufficient condition, not the necessary condition, is proved. *Cf.* also G. Salmon, *Higher Algebra*, 4th edit., §§63–64; and Gordan and Noether, in *Math. Annalen*, Vol. 10 (1876), pp. 549–552.

Proof. Consider any function $f(x_1, x_2, \ldots, x_n)$. Introducing the new variables $y_1, y_2, \ldots, y_n$, by the relations

$$\begin{aligned} x_1 &= y_1 + y_2 + \cdots + y_{n-1} + y_n, \\ x_2 &= \phantom{y_1 + {}} y_2 + \cdots + y_{n-1} + y_n, \\ &\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ x_{n-1} &= \phantom{y_1 + y_2 + \cdots + {}} y_{n-1} + y_n, \\ x_n &= \phantom{y_1 + y_2 + \cdots + y_{n-1} + {}} y_n, \end{aligned} \tag{2}$$

we have

$$f(x_1, x_2, \cdots, x_n) = F(y_1, y_2, \cdots, y_{n-1}, y_n);$$

whence

$$\frac{\partial F}{\partial y_n} = \frac{\partial f}{\partial y_n} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial y_n} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial y_n} + \cdots + \frac{\partial f}{\partial x_n}\frac{\partial x_n}{\partial y_n},$$

or, by (2),

$$\frac{\partial F}{\partial y_n} = \frac{\partial f}{\partial x_1} + \frac{\partial f}{\partial x_2} + \cdots + \frac{\partial f}{\partial x_n}. \tag{3}$$

Therefore f will satisfy the equation (1) when and only when $dF/dy_n = 0$; that is, when and only when F is a function of $y_1, y_2, \cdots, y_{n-1}$ alone. But from (2) we have at once,

$$y_1 = x_1 - x_2, \qquad y_2 = x_2 - x_3, \cdots, \qquad y_{n-1} = x_{n-1} - x_n;.$$

these differences are independent, a relation

$$x_i - x_{i+1} = F(x_1 - x_2, \cdots, x_{i-1} - x_i, \quad x_{i+1} - x_{i+2}, \cdots, x_{n-1} - x_n)$$

between them being impossible, since to give all the variables $x_1, x_2, \cdots, x_i$ the same increment, while $x_{i+1}, \cdots, x_n$ receive none, would lead to contradiction. Any other differences of the x's, as $x_i - x_j \, (j \neq i + 1)$, can however be expressed in terms of the former differences by the relation

$$x_i - x_j = y_i + y_{i+1} + \cdots + y_{j-1}.$$

Hence the theorem is true.*

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* The theorem in question is a special case of the following:

If $\phi_1, \phi_2, \cdots, \phi_{n-1}$ *are* $n - 1$ *independent solutions of the equation*

$$X_1\frac{\partial f}{\partial x_1} + X_2\frac{\partial f}{\partial x_2} + \cdots + X_n\frac{\partial f}{\partial x_n} = 0,$$

(*where the coefficients are any functions of the x's*), *then the necessary and sufficient condition that a function f be a solution of the equation is that f be a function of the ϕ's.* See, for example, E. Goursat, *Équations aux dérivées partielles du premier ordre* (1891), pp. 31–32.

It will be seen that this proof does not use the theory of determinants, or the theory of the Jacobian, nor is any knowledge of the properties of differential equations required.

If in equation (1) we replace x_i by x_i/a_i $(i = 1, 2, \cdot\cdot, n)$, we have the more general theorem that

A function $f(x_1, x_2, \cdot\cdot\cdot, x_n)$ *will be a solution of the equation*

$$a_1 \frac{\partial f}{\partial x_1} + a_2 \frac{\partial f}{\partial x_2} + \cdot\cdot\cdot + a_n \frac{\partial f}{\partial x_n} = 0, \tag{4}$$

when and only when f *is a function* of the $n-1$ independent differences $x_i/a_i - x_{i+1}/a_{i+1}$, or, more simply, *of the* $n-1$ *independent determinants* $a_{i+1}x_i - a_ix_{i+1}$.

For example, since the resultant, R, of two binary forms

$$a_0x^n + a_1x^{n-1} + \cdot\cdot\cdot + a_n = 0,$$
$$b_0x^n + b_1x^{n-1} + \cdot\cdot\cdot + b_n = 0,$$

satisfies the partial differential equation

$$b_0 \frac{\partial R}{\partial a_0} + b_1 \frac{\partial R}{\partial a_1} + \cdot\cdot\cdot + b_n \frac{\partial R}{\partial a_n} = 0,$$

it follows that R is a function of the $n-1$ independent determinants $a_ib_{i+1} - a_{i+1}b_i$.

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